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THE PROBLEM OF THE FLOW OF A PLASTIC MATERIAL ALONG A SURFACE*

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The equations of the perfect, rigid-plastic membrane theory of shells whose thickness is variable and unknown, are studied. The equations were studied earlier in /1-5/ for the case of a smooth surface of flow or for regular modes; singular modes are discussed below. It is shown that in the case of such modes and equations in question split into two systems. The first system yields the forces and the shell thickness, and after this the velocities are found from the second system. The Tresca flow conditions and the maximum reduced stress are studied as examples.

An approximate model of the flow of a plastic material over the surface of an instrument was constructed earlier in /1/. One-dimensional problems of the drawing and pressing of shells in axisymmetric matrices were studied in /2, 3/, and an iterative method of solving one-dimensional problems of this type was given in /4/. A number of later papers discussed the effect of material hardening /5/ and anisotropy /6/, and a model of a non-linearly viscous body was used in /7/ to study the hot deforming of shells.

We will use the equations of the membrane theory of shells written in a curvilinear orthogonal coordinate system: q_i $(i=1,2)$ are the coordinates of the principal lines of curvature and q_3 are the coordinates of the normal to the middle surface /1-8/.

$$(N_{11}H_2)_{,1}+(H_1N_{12})_{,2}+N_{12}H_{1,2}-N_{22}H_{2,1}-H_1H_2p_1=0 \tag{1}$$
$$(H_2N_{12})_{,1}+(H_1N_{22})_{,2}+N_{12}H_{2,1}-N_{11}H_{1,2}-H_1H_2p_2=0$$

$$\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}=p_3;\quad N_{ij}=h\sigma_{ij} \tag{2}$$
$$p_\alpha=\sigma_{3\alpha}\,|^{q_3}=h/2-\sigma_{3\alpha}\,|^{q_3}=-h/2,\quad \alpha=1,2,3$$

Here R_i are the principal radii of curvature of the middle surface, H_i are the Lame coefficients, and h is the thickness.

The deformation rates are defined as follows /1/:

$$\varepsilon_{11}=\frac{1}{H_1}v_{1,1}+\frac{v_2}{H_1H_2}H_{1,2},\quad \varepsilon_{22}=\frac{1}{H_2}v_{2,2}+\frac{v_1}{H_1H_2}H_{2,1} \tag{3}$$
$$2\varepsilon_{12}=\frac{H_1}{H_2}\left(\frac{v_1}{H_1}\right)_{,2}+\frac{H_2}{H_1}\left(\frac{v_2}{H_2}\right)_{,1}$$
$$\varepsilon_{33}=\frac{1}{h}\frac{dh}{dt}$$

In the case of a perfectly rigid-plastic material we assume that stresses σ_{ij} exist in three-dimensional space, of the following piecewise smooth flow surface:

$$F_n(\sigma_{ij})-k_n=0,\quad n=1,\ldots,m \tag{4}$$

(F_n are homogeneous, first-degree functions of the stresses) and the associated law of plastic flow (no summation over i) holds

$$\varepsilon_{ii}=\mu_n\frac{\partial F_n}{\partial\sigma_{ij}},\quad 2\varepsilon_{12}=\mu_n\frac{\partial F_n}{\partial\sigma_{12}} \tag{5}$$
$$\mu_n=0,\ \text{when}\ F_n<k_n\ \text{or}\ F_n=k_n,\ dF_n<0$$
$$\mu_n>0,\ \text{when}\ F_n=k_n\ \text{and}\ dF_n=0$$

The condition of incompressibility of the material, which follows from the associated law for the flow surfaces, depending on the stress tensor deviator components only, has the form

$$\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}=0 \tag{6}$$

When the friction p_1, p_2 is given and the geometry of the middle surface is such that it coincides, in what follows, with the surface of the instrument, then the system of equations (1)-(6) is closed with respect to the unkowns $N_{ij}, p_3, v_i, h, \mu_n$.

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Unlike the classical formulation of problem on the membrane plane-stress state of a plate and shells, the present model takes into account the change in the thickness which occurs during plastic flow, and this is essential in technological problems which deal with a developed plastic flow of a material. An analysis of the type of the system of equations for a smooth surface of flow or for the regular modes was carried out in /9, 10/.

Below we consider the following singular modes:

$$F_1 - k_1 = 0, \quad F_2 - k_2 = 0 \tag{7}$$

which are of interest in connection with the fact that flow surfaces, piecewise linear in the space of principal stresses σ_i, are often used when deriving the solution in closed form /2-4/.

From (7) we obtain the following expression for the forces, and the final relation for the thickness:

$$F_1(N_{ij}) - \beta F_2(N_{ij}) = 0 \tag{8}$$
$$h = \frac{F_1(N_{ij})}{k_1} \quad (\beta = k_1/k_2)$$

where F_n are homogeneous, first degree functions of the stresses.

The equations of equilibrium (1) and the first condition of (8) together form a closed system of equations for the forces, and the second equation of (8) is used to determine the thickness when the forces are known. Let us introduce the notation

$$a_{ijn} = \frac{\partial F_n}{\sigma\sigma_{ij}}$$
$$\Delta_1 = a_{111}a_{222} - a_{112}a_{221}$$
$$\Delta_2 = a_{121}a_{222} - a_{122}a_{221}$$
$$\Delta_3 = a_{111}a_{122} - a_{112}a_{121}$$

The associated law (5) yields the following relations for the deformation rates:

$$\Delta_1 2e_{12} = \Delta_2 e_{11} = \Delta_3 e_{22} \tag{9}$$
$$\mu_1 = (a_{222}e_{11} - a_{112}e_{22})/\Delta_1$$
$$\mu_2 = (a_{111}e_{22} - a_{221}e_{11})/\Delta_1$$

When the forces and the thickness h are known, Eq.(9) forms, together with the condition of incompressibility (6), a closed system for the velocities v_i, unlike the plane problem disregarding the thickness /11/ where the singular mode leads to a kinematically indeterminate problem.

The type of the system of equations for the forces depends on the sign of $B = b_{12}^2 - 4b_{11}b_{22}$ where $b_{ij} = a_{ij1} - \beta a_{ij2}$. If $B > 0$, then the system of equations is hyperbolic, and the equations for the characteristics and the relations on them have the form

$$\frac{dq_2}{dq_1} = \lambda_m = \frac{H_1}{H_2}(-b_{12} \pm \sqrt{B})/2b_{22}, \quad m = 1, 2$$
$$b_{11}H_1^2H_2dN_{11} - b_{12}H_1H_2^2\lambda_m dN_{12} + [b_{11}H_1^2(H_{2,1}(N_{11} - N_{22}) +$$
$$2N_{12}H_{1,2} - H_1H_2p_1) - \lambda_m b_{12}H_1H_2(H_{1,2}(N_{11} - N_{22}) + 2N_{12}H_{2,1} -$$
$$H_1H_2p_2)],\ dq_1 = 0$$

With $B = 0$, the force equations are parabolic and we have a unique characteristic

$$\frac{dq_2}{dq_1} = \lambda = -\frac{H_1b_{12}}{H_2^2b_{22}}$$
$$H_2dN_{11} + [H_{2,1}(N_{11} - N_{22}) + 2N_{12}H_{1,2} - p_1]\,dq_1 = 0$$
$$H_2dN_{12} + [H_{1,2}(N_{11} - N_{22}) + 2N_{12}H_{2,1} - p_2]\,dq_1 = 0$$

If $B < 0$, then the force equations are elliptic.

Let us consider a system of equations for the velocities when the thickness h and the stresses σ_{ij} are known

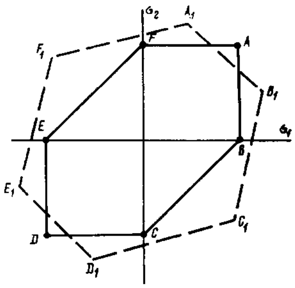
$$\Delta_1 2\varepsilon_{12} = \Delta_2\varepsilon_{11} + \Delta_3\varepsilon_{22}, \quad \varepsilon_{11} + \varepsilon_{22} = -\varepsilon_{33} \tag{10}$$

The Eqs.(10) are hyperbolic with respect to the velocities v_i, and the characteristics and the relations on them have the form

$$\frac{dq_2}{dq_1} = \lambda_m = \frac{H_1}{H_2}(\Delta_2 - \Delta_3 \pm \sqrt{(\Delta_2 - \Delta_3)^2 + 4\Delta_1^2})/2\Delta_1$$
$$H_1^2H_2\,dv_1 + \lambda_m H_1H_2^2\,dv_2 + \Big[v_2H_1^2H_{1,2} + \lambda_m^2H_2^2H_{2,1}v_1 - \lambda_m H_1H_2H_{1,2}v_1 -$$
$$\lambda_m H_1H_2H_{2,1}v_2 + H_1^3H_2^2e_{33}\left(\frac{1}{H_2} + \frac{\lambda_m\Delta_2}{H_1\Delta_1}\right)\Big]\,dq_1 = 0$$

In what follows we shall restrict ourselves, for simplicity, to the case of the Cartesian coordinate system ($H_1 = H_2 = 1$) and assume that $p_1 = p_2 = 0$. It is then conditions of flow

$F_n = a_{in}\sigma_i - k_n$, $a_{in} = \mathrm{const}$, piecewise linear in the space of principal stresses σ_i that are considered most often. The singular mode corresponds to the point $\sigma_1 = \sigma_1^*$, $\sigma_2 = \sigma_2^*$ in the plane (see the figure). We obtain $N_1 - \beta N_2 = 0$, $\beta = \sigma_1^*/\sigma_2^*$ for the forces N_i.



Using well-known formulas, we write the expression for the forces N_{ij} in terms of the principal forces

$$N_{11} = p + t\cos 2\varphi, \quad N_{22} = p - t\cos 2\varphi \tag{11}$$
$$N_{12} = t\sin 2\varphi$$
$$p = \frac{N_1 + N_2}{2}, \quad t = \frac{N_1 - N_2}{2}$$

Here φ is the angle between the direction of N_1 and the Ox_1 axis. The functions p and t are connected by the linear relation

$$p = \varkappa t, \quad \varkappa = \frac{\beta + 1}{\beta - 1} \tag{12}$$

Thus the system of equilibrium equations and conditions (11, (12) yield a system of equations for the unknowns t, φ

$$\begin{aligned} &N_{11} = t(\varkappa + \cos 2\varphi), \quad N_{22} = t(\varkappa - \cos 2\varphi)\\ &N_{12} = t\sin 2\varphi\\ &t_{,1}(\varkappa + \cos 2\varphi) - 2t\sin 2\varphi\varphi_{,1} + t_{,2}\sin 2\varphi + 2t\cos 2\varphi\varphi_{,2} = 0\\ &t_{,1}\sin 2\varphi + 2t\cos 2\varphi\varphi_{,1} + t_{,2}(\varkappa - \cos 2\varphi) + 2t\sin 2\varphi\varphi_{,2} = 0 \end{aligned} \tag{13}$$

The type of Eqs.(13) is determined by the sign of $B = \varkappa^2 - 1$. If $B > 0$, then the equations are hyperbolic and the characteristics and relations on them have the form

$$\frac{dx_2}{dx_1} = \lambda_m = \frac{\varkappa\sin 2\varphi \pm \sqrt{\varkappa^2 - 1}}{1 + \varkappa\cos 2\varphi}$$
$$d\eta_m = \sqrt{\varkappa^2 - 1}\, d\ln t \pm d\varphi = 0$$

Choosing η_m as the new independent variables, we obtain a linear system of equations in canonical form

$$\frac{\partial x_2}{\partial \eta_2} = \lambda_1\frac{\partial x_1}{\partial \eta_2}, \quad \frac{\partial x_2}{\partial \eta_1} = \lambda_2\frac{\partial x_1}{\partial \eta_1}$$

The transformation is possible, provided that the Jacobian

$$J = \frac{\partial x_2}{\partial \eta_1}\frac{\partial x_2}{\partial \eta_2}(\lambda_2 - \lambda_1) = \frac{\partial x_1}{\partial \eta_1}\frac{\partial x_1}{\partial \eta_2}(\lambda_1 - \lambda_2) \neq 0$$

The case of $J = 0$ yields simple integrals. If $d\eta_1 = d\eta_2 = 0$, then $\varphi = \mathrm{const}$, $t = \mathrm{const}$ and the characteristics are straight lines. When $d\eta_1 = 0$, we find that the characteristics of the second family are rectilinear and $d\ln t = d\varphi = 0$ along them, i.e. $x_2 - \lambda_2 x_1 = f(\varphi)$. If $\varkappa^2 = 1$, then Eqs.(13) are parabolic and the characteristics coincide with the direction of one of the principal stresses

$$\frac{dx_2}{dx_1} = \mathrm{tg}\,\varphi, \quad \varkappa = 1, \quad \frac{dx_2}{dx_1} = -\mathrm{ctg}\,\varphi, \quad \varkappa = -1$$

In this case Eqs.(13) will be transformed to

$$\varphi_{,1}\varkappa\sin 2\varphi + \varphi_{,2}(1 - \varkappa\cos 2\varphi) = 0$$
$$t_{,1}\varkappa\sin 2\varphi + t_{,2}(1 - \varkappa\cos 2\varphi) + 2t\varkappa(\cos 2\varphi\varphi_{,1} + \sin 2\varphi\varphi_{,2}) = 0$$

The solution of this equation is given in terms of two arbitrary functions:

$$x_2 - \lambda x_1 = f_1(\varphi), \quad t = \frac{f_2(\varphi)}{2x_1 + (1 - \varkappa\cos 2\varphi) f_1'(\varphi)}$$

We note that if the singular mode appears in the first and third quadrant (see the figure) then the force equations are hyperbolic, while for the second and fourth quadrant Eqs.(13) are elliptic. The coordinate lines cooresspond to the parabolic force equations.

With regard to the velocities v_i we note that the characteristics in this case are the lines of greatest tangential stresses. Indeed,

$$\Delta_1 = 2d\cos 2\varphi, \quad \Delta_2 = -\Delta_3 = 2d\sin 2\varphi$$
$$L_m = \mathrm{tg}(\varphi \pm \pi/4), \quad d = a_{11}a_{22} - a_{12}a_{21}$$

Thus, if the stress field is known, the field of characteristics for the system of equations in velocities (10) is constructed automatically.

We shall consider, as examples, the Tresca flow conditions (the solid lines in the figure), and the maximum reduced stress (the dashed lines), most often encountered in the literature /11/. The points F, B, C, E lead to a system of parabolic force equations, and the points A_1, B_1, D_1, E_1 lead to hyperbolic-type equations. In case of the modes A, D, we obtain the simplest stress field $\sigma_{11} = \sigma_{22} = \pm 2k$, $\sigma_{12} = 0$ and $h = \text{const}$. In this case $\Delta_1 = 0, \Delta_2 = \Delta_3 = 0$ and since $\varepsilon_{33} = \frac{d}{dt} \ln h \equiv 0$, we have a unique expression $\varepsilon_{11} + \varepsilon_{22} = 0$ for the velocities, i.e. we obtain an indeterminate problem for the velocities just as in the plane problem ignorning the variation in thickness. We note that in the case of the modes A, D the dissipation of plastic energy $W = \sigma_{ij}\varepsilon_{ij} \equiv 0$, and this can only correspond to a rigid region. The points F_1, C_1 correspond to the elliptic force equations and lead to the condition

$$N_{11} + N_{22} = 0 \tag{14}$$

Let us introduce the force function satisfying identically the equations of equilibrium (1) when $p_1 = p_2 = 0$:

$$N_{11} = \frac{1}{H_2}\left(\frac{1}{H_2}\Phi_{,2}\right)_{,2} + \frac{H_{2,1}}{H_1^2 H_2}\Phi_{,1}, \quad N_{12} = -\frac{1}{H_1}\left(\frac{1}{H_2}\Phi_{,2}\right)_{,1} + \frac{H_{1,2}}{H_1^2 H_2}\Phi_{,1}$$

$$N_{22} = \frac{1}{H_1}\left(\frac{1}{H_1}\Phi_{,1}\right)_{,1} + \frac{H_{1,2}}{H_1 H_2^2}\Phi_{,2}$$

Taking into account conditions (14), we obtain a Laplace equation for the force function Φ. Its solution for the shell of revolution $q_1 = z;\ q_2 = \theta,\ H_{1,2} = H_{2,2} = 0$ can be sought in the form of an expansion in Fourier series

$$\Phi = \sum_{n=0}^{\infty} \Phi_{1n} \sin n\theta + \Phi_{2n} \cos n\theta$$

in which case we have

$$\frac{\partial^2 \Phi_{jn}}{\partial \bar{q}_1^2} + \frac{H_{2,1}}{H_1 H_2}\frac{\partial \Phi_{jn}}{\partial \bar{q}_1} - \frac{n^2}{H_2^2}\Phi_{jn} = 0, \quad j = 1, 2 \tag{15}$$

$$d\bar{q}_1 = H_1 dq_1$$

For a circular plate the solution of (15) has the form

$$\Phi = \sum_{n=0}^{\infty} (a_n r^n + b_n r^{-n}) \sin n\theta + (c_n r^n + d_n r^{-n}) \cos n\theta$$

$$(a_n, b_n, c_n, d_n = \text{const})$$

When $N_{12} = v_2 = 0$, we obtain the following expression for an annular plate:

$$\sigma_{11} = -\sigma_{22} = \pm k; \quad h = \frac{c_1}{r^2}, \quad v_1 = c_2 r - c' c^{-1} r \ln r$$

From the requirement that the energy dissipation is positive in the plastic region, we have

$$W = \sigma_{ij}\varepsilon_{ij} = \pm k c' c^{-1} > 0$$

Thus even the simplest example shows that when the boundary conditions and plate thickness are specially chosen, a singular mode can be obtained. In more complex, technological cases, whole regions may correspond to a singular mode. Hence, when dealing with the problems of a plane stress state and varying thickness, we should solve the problem of conjugation for the piecewise smooth flow surfaces not only for the regular, but also for the singular modes.

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EFFECTIVE PROPERTIES OF MULTICOMPONENT ELASTOPLASTIC COMPOSITE MATERIALS*

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The present paper generalizes the results obtained in /1/ to the case of an arbitrary number of elastic and elastoplastic components of the medium, by considering the elastoplastic behaviour of a multicomponent composite materials (CM).

1. Consider an elastoplastic, microinhomogeneous medium consisting of n different isotropic components joined to each other with perfect adhesion. Let the first m components be elastoplastic, and the remaining $n-m$ components be perfectly elastic. Hooke's law for such a CM has the form

$$\sigma_{ij}^{(s)} = 2\mu_s(\varepsilon_{ij}^{(s)} - e_{ij}^{p(s)}) + \delta_{ij}\lambda_s\varepsilon_{pp}^{(s)} \ (s=1,2,\ldots,m) \tag{1.1}$$

$$\sigma_{ij}^{(s)} = 2\mu_s\varepsilon_{ij}^{(s)} + \delta_{ij}\lambda_s\varepsilon_{pp}^{(s)} \quad (s=m+1,\ m+2,\ldots,n)$$

Here σ_{ij}, ε_{ij}, e_{ij}^p are the components of the stress, total and plastic deformation tensors, μ_s, λ_s are the Lamé parameters of the component materials, and the plastic deformations satisfy the condition of incompressibility $e_{pp}{}^p = 0$. The plastic properties of the elastoplastic components are described in terms of the Mises yield surface (k_s are the yield points)

$$s_{ij}s_{ij} = k_s^2 \ (s=1,2,\ldots,m), \quad s_{ij} = \sigma_{ij} - {}^1/_3\delta_{ij}\sigma_{pp}$$

The structure of CM can be described by a set of random indicator functions of the coordinates $\varkappa_1(\mathbf{r}), \varkappa_2(\mathbf{r}),\ldots,\varkappa_n(\mathbf{r})$. Every one of these functions $\varkappa_s(\mathbf{r})$ is equal to unity on the set of points of the s-th component, and to zero outside this set. Using these functions we can write the local Hooke's law in the form

$$\sigma_{ij}(\mathbf{r}) = 2\mu(\mathbf{r})(\varepsilon_{ij}(\mathbf{r}) - e_{ij}{}^p(\mathbf{r})) + \delta_{ij}\lambda(\mathbf{r}))\,\varepsilon_{pp}(\mathbf{r}) \tag{1.2}$$

where

$$\mu(\mathbf{r}) = \sum_{s=1}^{n}\mu_s\varkappa_s(\mathbf{r}), \quad \lambda(\mathbf{r}) = \sum_{s=1}^{n}\lambda_s\varkappa_s(\mathbf{r})$$

$$\varkappa_s(\mathbf{r})\,e_{ij}^p(\mathbf{r}) \equiv 0 \quad (s=m+1,\ m+2,\ldots,n)$$

All functions $\varkappa_s(\mathbf{r})$, stress tensors, total and plastic deformation tensors are assumed to be statistically homogeneous and ergodically random fields, and their expectations are replaced by the following quantities /2/ averaged over the component volumes V_s and over the whole volume V of the medium:

$$\langle f\rangle = \frac{1}{V}\int_V f(\mathbf{r})\,d\mathbf{r},\ \langle f\rangle_s = \frac{1}{V_s}\int_{V_s} f(\mathbf{r})\,d\mathbf{r} \ \left(V = \sum_{s=1}^{n}V_s\right)$$

Supplementing relation (1.2) with the equations of equilibrium $\sigma_{ij,j}(\mathbf{r}) = 0$ and the Cauchy formulas $2\varepsilon_{ij}(\mathbf{r}) = u_{i,j}(\mathbf{r}) + u_{j,i}(\mathbf{r})$ connecting the components of the total deformation tensor with the components of the displacement vector $u_i(\mathbf{r})$, we obtain a closed system of equations describing the deformation of a multicomponent CM whose boundary conditions are that there are no fluctuations in the value of the quantities on the surface S of the volume V

$$f(\mathbf{r})|_{\mathbf{r}\in S} = \langle f\rangle$$

Using Green's tensor

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